Supersymmetric quantum mechanics based on higher excited states II: a few new examples of isospectral partner potentials

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Abstract. We apply the generalized formalism and the techniques of the supersymmetric (susy) quantum mechanics to the cases where the super-potential is generated/defined by higher excited eigenstates (Robnik 1997, paper I). The generalization is technically almost straightforward but physically quite nontrivial since it yields an infinity of new classes of susy-partner potentials, whose spectra are exactly identical except for the lowest $n + 1$ states, if the superpotential is defined in terms of the $(n + 1)$-st eigenfunction, with $n = 0$ reserved for the ground state. First we show that there are practically no possibilities for shape invariant potentials based on higher excited states. Then we calculate the isospectral partner potentials for the following 1-dim potentials (after separation of variables where appropriate): (i) 3-dim (spherically symmetric) harmonic oscillator, (ii) 3-dim (isotropic) Kepler problem, (iii) Morse potential, (iv) Pöschl-Teller type I potential, and (v) the 1-dim box potential. In all cases except in (v) we get new classes of solvable potentials. In (v) the partner potential to the box potential is a special case of Pöschl-Teller type I potential.

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1 Introduction

In a recent paper (Robnik 1997, paper (**I**)) it has been shown that the formalism of the supersymmetric (nonrelativistic) quantum mechanics can be applied also to the higher excited states (say, n-th state) of 1-dim potentials, generating new partner potentials isospectral to the original potential, except for the lowest $n+1$ states which are simply just missing. There we gave the example of the 1-dim harmonic oscillator, which for all $n > 0$ yields new classes of rational potentials. In this paper we present results of a straightforward further application of this formalism to a few most important exactly solvable 1-dim potentials, namely (i) spherically symmetric 3-dim harmonic oscillator, (ii) 3-dim isotropic (spherically symmetric) Kepler potential, (iii) Morse potential, (iv) Pöschl-Teller type I potential, and (v) 1-dim box potential.

Following the seminal papers of Witten (1981) and Gendenshtein (1983) the methods of supersymmetric (susy) (nonrelativistic) quantum mechanics have quickly developed and it has been realized, that (1) there exist partner potentials with precisely the same energy spectra except for the ground state ($n = 0$) (whose wavefunction $\phi(x) = \psi_0(x)$ is used to generate/define the superpotential $W(x)$ - see below) [3], and that (2) if they are *"shape invariant"*, their spectra and wavefunctions can be exactly and analytically solved. It is believed that the list of such susy-0 shape invariant partner potentials is now complete and finite (Lévai 1989, Barclay *et al* 1993), and therefore quite limited in use. The research has been later further developed also in direction of applying the WKB methods to such classes of Hamiltonians, including the search for improved simple quantization conditions which would be exact in case of susy shape invariant potentials (Barclay, Khare and Sukhatme 1993, Barclay and Maxwell 1991, Barclay 1993, Inomata, Junker and Suparmi 1993, Junker 1995, Robnik and Salasnich 1997), and also in direction of exploring the applicability of the path integral techniques (Inomata and Junker 1991,1994). One of the nicest presentations of susy quantum mechanics was published by Dutt, Khare and Sukhatme (1988), henceforth referred to as DKS. We will use their notations. It should be mentioned at this place

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[3] the ground state energy E_0^- is missing in the partner Hamiltonian H^+, so that its groundstate $E_0^+ = E_1^-$

that the ideas involved behind the susy property and shape invariance were formulated first by Infeld and Hull (1951), where they were called the "factorization method", and these authors refer further to the related ideas in the works of Schrödinger (1940,1941).

Thus we shall use the notations of DKS, employed also in (I), and present the brief outline of the susy-m formalism in the section 2. Then we demonstrate in section 3 that there are practically no cases of susy-m shape invariance for $n > 0$. In sections 4-8 we present the partner potentails of (i) through (v), and in section 9 we draw the general conclusions and discuss the results.

2 Generalized supersymmetric formalism

The main point of this section is to briefly review the formalism of the susy-n quantum mechanics, which thus can be generalized to arbitrary higher excited eigenstates $\phi(x) = \psi_n(x)$, $n = 0, 1, 2, \ldots$, used to generate the superpotential $W(x)$, namely

$$W(x) = -\frac{\hbar}{\sqrt{2\mu}} \frac{\phi'}{\phi},$$ (1)

where $\phi'(x) = d\phi/dx$, μ is the mass of the particle moving in the V^- potential, $2\pi\hbar$ is the Planck constant and n is the quantum number equal to the number of nodes of the eigenfunctions $\psi_n(x)$ of the starting potential $V^-(x)$. The energy scale is adjusted so that the $(n+1)$-st energy eigenvalue is exactly zero, $E_n^- = 0$. The corresponding Hamiltonian is $H^- = -\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + V^-(x)$, and the Schrödinger equation reads

$$H^-\psi_n^- = H^-\phi = (-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + V^-(x))\phi = 0.$$ (2)

Obviously, because $\phi'(x) \neq 0$ at the nodes y_j, the superpotential $W(x)$ will have singularities at the nodes y_j, $j = 1, 2, \ldots, n$ of ϕ. However, this does not invalidate our derivation, but it merely means, as will become clear later on, that the partner potential generated by ϕ diverges to $+\infty$ when $x \to y_j$, for any $j = 1, 2, \ldots, n$. This implies that the potential wells are well defined between two consecutive singularities and that they do not communicate with solutions in the neighbouring wells. Thus if $n = 0$ we have the common case

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of usual susy potentials defined on $(-\infty, +\infty)$, if $n = 1$ we have two separated potential wells, each of them on a semiinfinite domain, for $n = 2$ we have one infinite potential well on a finite domain between two nodes y_1 and y_2, and two binding potential wells on the two semiinfinite domains $(-\infty, y_1]$ and $[y_2, +\infty)$, and so on. The (partner) potentials constructed in this way are nontrivial and certainly very interesting since they contribute to our list of solvable potentials which now becomes truly very rich and infinite in its contents.

In order to make this paper selfcontained we will build up the formalism necessary to construct the partner potentials and to define the shape invariance, following DKS, in order to demonstrate that the susy formalism does not break down anywhere on its domain of definition, and to define the language needed to talk about further results that we shall present in this contribution.

First we express the starting potential $V^-(x)$ in terms of the $(n + 1)$-st eigenfunction $\phi(x) = \psi_n(x)$, by solving (2)

$$V^-(x) = \frac{\hbar^2}{2\mu} \frac{\phi''}{\phi},\tag{3}$$

which is regular everywhere, because at the nodes y_j the second derivative $\phi''(x) = d^2\phi/dx^2$ also vanishes with ϕ. Thus the basic Hamiltonian H^- reads

$$H^- = \frac{\hbar^2}{2\mu}\left(-\frac{d^2}{dx^2} + \frac{\phi''}{\phi}\right).\tag{4}$$

The two important operators are:

$$A^\dagger = \frac{\hbar}{\sqrt{2\mu}}\left(-\frac{d}{dx} - \frac{\phi'}{\phi}\right),\tag{5}$$

and

$$A = \frac{\hbar}{\sqrt{2\mu}}\left(\frac{d}{dx} - \frac{\phi'}{\phi}\right),\tag{6}$$

which gives

$$H^- = A^\dagger A.\tag{7}$$

We further *define the partner Hamiltonian H^+ and the partner potential V^+* as

$$H^+ = AA^\dagger = -\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + V^+(x), \tag{8}$$

where

$$V^+(x) = V^-(x) - \frac{\hbar^2}{\mu}\frac{d}{dx}(\frac{\phi'}{\phi}) \tag{9}$$

or

$$V^+(x) = -V^-(x) + \frac{\hbar^2}{\mu}(\frac{\phi'}{\phi})^2. \tag{10}$$

The potentials V^+ and V^- are called *susy-n partner potentials*. We will show that they have the same energy levels, except for the $(n+1)$ lowest states of V^- for which there are no corresponding states of V^+, so that the ground state of the latter one is $E_0^+ = E_{n+1}^-$. All higher states have then identical energies. From equation (10) we see explicitly that at every node y_j, $j = 1, 2, \ldots, n$ of the defining eigenstate $\phi = \psi_n^-$ the partner potential(s) will have a singularity of the type $1/(x - y_j)^2$ such that $V^+(x) \to +\infty$ when $x \to y_j$, so that every branch of the partner potential will be confining up to infinity, and the solutions in various branches do not communicate. Thus for each n we shall find $(n+1)$ (branches of the) partner potentials.

In terms of the superpotential W defined in equation (1) we can write

$$\phi(x) = \psi_n^-(x) = \exp(-\frac{\sqrt{2\mu}}{\hbar}\int^x W(x)dx), \tag{11}$$

which is well defined in the definition domain of any of the branches of the partner potential, and obviously ϕ will go to zero where W has the poles $1/(x - y_j)$, as it should happen.

Some of the relationships can be rewritten/reformulated in terms of the superpotential $W(x)$ now:

$$A^\dagger = -\frac{\hbar}{\sqrt{2\mu}}\frac{d}{dx} + W(x),$$

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$$A = \frac{\hbar}{\sqrt{2\mu}}\frac{d}{dx} + W(x). \tag{12}$$

Further we observe,

$$V^{\pm}(x) = W^2(x) \pm \frac{\hbar}{\sqrt{2\mu}}W'(x), \qquad W'(x) = \frac{dW}{dx}, \tag{13}$$

and also

$$V^+ = V^- + \frac{2\hbar}{\sqrt{2\mu}}\frac{dW}{dx}. \tag{14}$$

The commutator of the operators A and $A^\dagger$ is

$$[A, A^\dagger] = \frac{2\hbar}{\sqrt{2\mu}}\frac{dW}{dx}. \tag{15}$$

Now we have all tools at hand to show that the susy partner potentials V^- and V^+ are isospectral except for the lowest $(n+1)$ states of V^- which have no counterpart in V^+, so that its ground state is $E_0^+ = E_{n+1}^-$.

The demonstration, following DKS, is very easy: First we find that if ψ_m^- is an eigenfunction of H^- with the eigenenergy E_m^-, then $A\psi_m^-$ is an eigenfunction of H^+ with the same energy:

$$H^+(A\psi_m^-) = AA^\dagger A\psi_m^- = AH^-\psi_m^- = AE_m^-\psi_m^- = E_m^- A\psi_m^-. \tag{16}$$

Now we show that this applies only to the eigenstates m higher than n, $m = n+1, n+2, \ldots$, by considering the normalization condition, by writing the normalized state $\psi_m^+ = C_m A\psi_m^-$, and calculating the normalizing coefficient C_m,

$$\|\psi_m^+\|^2 = C_m^2 < A\psi_m^-|A\psi_m^- >= C_m^2 < \psi_m^-|A^+A\psi_m^- >= C_m^2 E_m^-\|\psi_m^-\|^2. \tag{17}$$

If all ψ_m^- are normalized (they are certainly orthogonal, because we deal with one dimensional systems, where degeneracies are forbidden due to the Sturm-Liouville theorem (Courant and Hilbert 1968) and therefore all eigenstates must be orthogonal), then

$$C_m = \frac{1}{\sqrt{E_m^-}}, \tag{18}$$

which implies that the construction succeeds only iff $E_m^- > 0$, implying that $m > n$. Thus the two Hamiltonians H^- and H^+ defined in (4) and in (8) are isospectral except for the lowest $(n+1)$ eigenstates of H^- which have no counterpart in H^+.

Counting now the eigenstates of H^+ from $m = 0, 1, 2, \ldots$, where $m = 0$ is the ground state, and m is the number of nodes of the (now also normalized) eigenfunction ψ_m^+, we have

$$\psi_m^+ = \frac{1}{\sqrt{E_{n+1+m}^-}} A\psi_{n+1+m}^-, \qquad E_m^+ = E_{n+1+m}^-. \tag{19}$$

Of course it is easy to show that, conversely, for every eigenstate ψ_m^+ of H^+ there exists the normalized eigenstate of H^-, namely

$$\psi_{n+1+m}^- = \frac{1}{\sqrt{E_m^+}} A^\dagger \psi_m^+, \qquad m = 0, 1, 2, \ldots \tag{20}$$

This completes our proof of isospectrality, generalized to the case that the generating function ϕ of the superpotential W, defined in equation (1), is a higher excited wavefunction, namely $\phi = \psi_n^-$, $n = 0, 1, 2, \ldots$. As we have seen, the formalism of superpotential and of the partner potentials works everywhere except at the singularities located at the nodal points y_i of ϕ, where the partner potential V^+ goes to infinity as $1/(x - y_i)^2$, thereby defining several branches of V^+ well defined on their disjoint domains of definition.

We have demonstrated that if one of the partner systems (the Hamiltonians) can be solved completely (by calculating the energy levels and the eigenfunctions), then the susy formalism enables one to solve the partner problem completely, following equation (19). One of the most important cases is of course the harmonic oscillator, which has been discussed in (I), whilst in this paper we study five more examples (i)-(v) announced in Introduction.

3 Is there shape invariance for $n > 0$?

If the solutions for the two partner Hamiltonians are both unknown, then another approach is necessary to solve them. In case of the standard susy formalism with $n = 0$ we have the important class of the shape invariant potentials. As is well known (DKS) the shape invariance of the two partner potentials V^- and V^+ is defined by

$$V^+(x; a_0) = V^-(x; a_1) + R(a_1), \tag{21}$$

where a_0 is a set of parameters, $a_1 = f(a_0)$ and $R(a_1)$ is independent of x. The procedure is now (essentially embodied in the factorization method of Infeld and Hull (1951)) the following. Consider a series of Hamiltonians $H^{(s)}$, $s = 0, 1, 2, \ldots$, where $H^{(0)} = H^-$ and $H^{(1)} = H^+$, by definition

$$H^{(s)} = -\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + V^-(x; a_s) + \sum_{k=1}^{s} R(a_k), \tag{22}$$

where

$$a_s = f^s(a_0) = \underbrace{f \circ \ldots \circ f}_{s}(a_0). \tag{23}$$

Now compare the spectra of $H^{(s)}$ with $H^{(s+1)}$, and find

$$
\begin{aligned}
H^{(s+1)} &= -\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + V^-(x; a_{s+1}) + \sum_{k=1}^{s+1} R(a_k), \\
H^{(s+1)} &= -\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + V^+(x; a_s) + \sum_{k=1}^{s} R(a_k),
\end{aligned}
\tag{24}
$$

Thus it is obvious that $H^{(s)}$ and $H^{(s+1)}$ are susy partner Hamiltonians, and they have the same spectra from the first level upwards except for the ground state of $H^{(s)}$ whose energy is

$$E_0^{(s)} = \sum_{k=1}^{s} R(a_k). \tag{25}$$

When going back from s to $(s - 1)$ we reach $H^{(1)} = H^+$ and $H^{(0)} = H^-$, whose ground state energy is zero and its m-th energy level being coincident

with the ground state of the Hamiltonian $H^{(m)}$, $m = 1, 2, \ldots$. Therefore the complete spectrum of H^- is

$$E_m^- = \sum_{k=1}^{m} R(a_k), \qquad E_0^- = 0. \tag{26}$$

The generalization of shape invariance to the case of any $n \geq 0$ is straightforward, but it results in higher complexity and therefore it is more rarely satisfied by the specific systems. By repeating the above argumentation we reach the conclusion that, when (21) is satisfied for a superpotential W with given n, then we cannot calculate the entire spectrum of the shape invariant potential/Hamiltonian H^-, but only the subset (subsequence) of period $n + 1$, namely

$$E_{n+m(n+1)}^- = \sum_{k=1}^{m} R(a_k), \qquad E_n^- = 0, m = 1, 2, \ldots. \tag{27}$$

In the special case $n = 0$ we of course recover the formula (26). For $n > 0$ we have *none example* of susy-n shape invariance so far. In fact we shall show now that there is no susy-n shape invariance for $n > 0$, unless we find some rare exceptions.

In fact we can see in equation (21) that if $V^-(x; a_1)$ has no singularities as a function of x, then the partner potential $V^+(x; a_0)$ also has no singularities, because $R(a_1)$ is just a constant and independent of x. Therefore, for $n > 0$ a potential $V(x)$ cannot be shape invariant if it has no singularities. The only possibility then is that $V^-(x; a_1)$ has singularities at the same places y_i as the partner potential $V^+(x; a_0)$: This, however seems also impossible, because for $n > 0$ the partner potential V^+ obtains new singularities between those of V^-, corresponding to the nodes of the n-th eigenfunction of V^-. Therefore, unless we find some pathological exceptions, it seems that there is no shape invariance for higher excited states $n > 0$, at least not in the sense of the definition (21). Perhaps some other functional relationships yet to be discovered might lead to some other type of "shape invariance".

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4 The 3-dim spherically symmetric harmonic oscillator

Let us consider a few examples of susy-m partner potentials, first the 3-dim spherically symmetric harmonic oscillator. (The case of 1-dim harmonic oscillator was calculated and discussed in (I).) To prepare some generalities of 3-dim spherically symmetric potentials $V(r)$ we first write down the Schrödinger equation $\hat{H}\psi = E\psi$, namely

$$[-\frac{\hbar^2}{2\mu}(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}) + \frac{\hat{L}^2}{2\mu r^2} + V(r)]\psi = E\psi, \tag{28}$$

where μ is the mass of the particle and $\hat{H}, \hat{L}^2$ and $\hat{L}_z$ are the usual notations for the Hamilton operator, (square of the) angular momentum, and the z-component of the angular momentum. The quantum numbers of the latter two will be denoted by l and m, so that $l = 0, 1, 2, \ldots$ and $m = l, l - 1, l - 2, \ldots, -(l-1), -l$, and the eigenvalues of $\hat{L}^2$ are $l(l+1)\hbar^2$. Due to the spherical symmetry we have thus the separation of variables

$$\psi(r, \theta, \varphi) = R(r)Y_{lm}(\theta, \varphi), \tag{29}$$

where Y_{lm} are the spherical harmonics, so that

$$[-\frac{\hbar^2}{2\mu}(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}) + V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}]R = ER, \tag{30}$$

and after the substitution

$$R(r) = \chi(r)/r, \qquad V_l(r) = V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}, \tag{31}$$

the radial Schrödinger equation becomes finally

$$[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_l(r)]\chi(r) = E\chi(r), \tag{32}$$

which is now just a 1-dim problem with the effective potential $V_l(r)$ defined in equation (31), for the physical range of definition $r \geq 0$. From equation (29) follows the normalization condition

$$\int_0^\infty dr \chi^2(r) = 1. \tag{33}$$

Now we look at the specific case of the 3-dim harmonic oscillator defined by

$$V(r) = \frac{1}{2}\mu\omega^2 r^2. \tag{34}$$

Introducing the constants

$$k^2 = \frac{2\mu E}{\hbar^2}, \qquad \lambda^2 = \frac{\mu\omega}{\hbar}, \tag{35}$$

we rewrite the Schrödinger equation (32) as

$$\frac{d^2\chi}{dr^2} + [k^2 - \lambda^2 r^2 - \frac{l(l+1)}{r^2}]\chi = 0, \tag{36}$$

with the solutions

$$\chi_n(r) = Cr^{l+1}e^{-\frac{1}{2}\lambda^2 r^2}F(-n, l + \frac{3}{2}; \lambda^2 r^2), \tag{37}$$

where n is the radial quantum number $n = 0, 1, 2, \ldots$ (in denoting the eigenfunctions and eigenvalues we shall suppress the quantum number l for brevity of notation), and the energy eigenvalues read

$$E_n = (2n + l + \frac{3}{2})\hbar\omega. \tag{38}$$

In the above $F(a, b; x)$ is the confluent hypergeometric series, [4] and the normalization constant C in (37) is equal to (Goldhammer 1963)

$$C = [\frac{2^{l+2-n}(2l + 2n + 1)!!}{\sqrt{\pi}n![(2l + 1)!!]^2}]^{\frac{1}{2}}\lambda^{(l+3/2)}. \tag{39}$$

The superpotential defined according to (1) using the n-th eigenfunction (37) reads

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[4]It is also called Kummer's function and denoted by $M(a, b, z)$ e.g. in Abramowitz and Stegun (1965) p.504.

$$W_n(r) = -\frac{\hbar}{\sqrt{2\mu}}\{\frac{l+1}{r} - \lambda^2 r[1 + \frac{2nF(-n+1, l+5/2; \lambda^2 r^2)}{(l+3/2)F(-n, l+3/2; \lambda^2 r^2)}]\} \qquad (40)$$

so that for the susy-0 (ground state) superpotential we have

$$W_0(r) = \sqrt{\frac{\mu}{2}}\omega r - \frac{(l+1)\hbar}{\sqrt{2\mu}r}. \qquad (41)$$

Now we can calculate the partner potential V_n^+ starting from the shifted potential

$$V_n^-(r) = \frac{1}{2}\mu\omega^2 r^2 + \frac{l(l+1)\hbar^2}{2\mu r^2} - (2n+l+3/2)\hbar\omega \qquad (42)$$

where we see $E_n^- = 0$, and thus obtain the partner potential

$$
\begin{aligned}
V_n^+(r) &= V_n^-(r) + 2\frac{\hbar}{\sqrt{2\mu}}\frac{dW_n(r)}{dr} = \\
&= \frac{1}{2}\mu\omega^2 r^2 + \frac{(l+1)(l+2)\hbar^2}{2\mu r^2} - (2n+l+\frac{1}{2})\hbar\omega + I_n, \\
I_n &= \frac{2n\hbar\omega}{(l+3/2)F(0)}\{F(1) + \frac{2\mu\omega}{\hbar}r^2[\frac{(1-n)}{(l+5/2)}F(2) + \frac{n}{(l+3/2)}\frac{F^2(1)}{F(0)}]\}
\end{aligned}
$$

$$(43)$$

where we used the short-hand notation for the confluent hypergeometric function

$$F(i) = F(-n+i, l+\frac{3}{2}+i; \lambda^2 r^2), \qquad i = 0, 1, 2. \qquad (44)$$

It is readily seen that for $n = 0$ we get the shape invariant case of the two partner potentials

$$V_0^-(r, l) = \frac{1}{2}\mu\omega^2 r^2 + \frac{l(l+1)\hbar^2}{2\mu r^2} - (l+\frac{3}{2})\hbar\omega$$

$$
\begin{aligned}
V_0^+(r, l) &= \frac{1}{2}\mu\omega^2 r^2 + \frac{(l+1)(l+2)\hbar^2}{2\mu r^2} - (l+\frac{1}{2})\hbar\omega \\
V_0^+(r, l) &= V_0^-(r, l+1) + 2\hbar\omega \\
V_0^+(r, l) &= V_0^-(r, a_1) + R(a_1), \\
a_0 &= l, \qquad a_1 = l+1, \qquad R(a_1) = 2\hbar\omega.
\end{aligned}
\tag{45}
$$

As the last point of this section we give the explicit expressions for the two lowest excited states ($n = 1, 2$):

$$
\begin{aligned}
V_1^+(r) &= \frac{1}{2}\mu\omega^2 r^2 + \frac{(l+1)(l+2)\hbar^2}{2\mu r^2} - (l+\frac{5}{2})\hbar\omega + I_1 \\
I_1 &= 2\beta_0\hbar\omega(1-z)^{-1}[1 + 2\beta_0 z(1-z)^{-1}] \\
\beta_0 &= (l+\frac{3}{2})^{-1}, \qquad z = \lambda^2 r^2 = \frac{\mu\omega}{\hbar}r^2
\end{aligned}
\tag{46}
$$

and

$$
\begin{aligned}
V_2^+(r) &= \frac{1}{2}\mu\omega^2 r^2 + \frac{(l+1)(l+2)\hbar^2}{2\mu r^2} - (l+\frac{9}{2})\hbar\omega + I_2 \\
I_2 &= 4\beta_0\hbar\omega(1 - 2\beta_0 z + \beta_0\beta_1 z^2)^{-1} \times \\
&\quad \times \{1 - \beta_1 z + 2z[-\beta_1 + 2\beta_0(1 - 2\beta_0 z + \beta_0\beta_1 z^2)^{-1}(1 - \beta_1 z)]\} \\
\beta_1 &= (l+\frac{5}{2})^{-1}, \qquad z = \lambda^2 r^2 = \frac{\mu\omega}{\hbar}r^2
\end{aligned}
\tag{47}
$$

The wavefunctions can be calculated using formula (19), where we have

$$
A = \frac{\hbar}{\sqrt{2\mu}}\frac{d}{dr} + W_n(r),
\tag{48}
$$

and using the eigenfunctions (37) and (40) we can write down the expression for the eigenfunctions of the susy-n partner potential, namely (bearing in mind $\chi_n^- = \chi_n$, from (37)),

$$
\chi_p^+ = \frac{1}{\sqrt{E_{n+1+p}^-}}A\chi_{n+1+p}^-,
\tag{49}
$$

with (using the equation (38))

$$E^-_{n+1+p} = E_{n+1+p} - E_n = 2(p+1)\hbar\omega, \qquad p = 0, 1, 2, \ldots \qquad (50)$$

5   The 3-dim spherically symmetric Kepler problem

The 3-dim spherically symmetric Kepler potential is

$$V(r) = -\frac{e^2}{r}. \qquad (51)$$

When using this in equations (28-32) we find the solutions

$$\chi_n(r) = Cr^{l+1}e^{-\frac{\lambda}{2}r}F(-n, 2l+2; \lambda r), \qquad (52)$$

where $n = 0, 1, 2, \ldots$ is the radial quantum number (counting the number of nodes of $\chi_n(r)$) and the energy eigenvalues are

$$E_n = -\frac{\mu e^4}{2\hbar^2(n+l+1)^2}, \qquad (53)$$

with the definitions of the parameters

$$\lambda = \frac{2}{(n+l+1)a}, \qquad a = \frac{\hbar^2}{\mu e^2}. \qquad (54)$$

The normalization condition is

$$\int_0^\infty \chi_n^2(r)dr = 1, \qquad (55)$$

so that the constant C is equal to

$$C = \frac{4}{a^{5/2}N^3(2l+1)!}\sqrt{\frac{(N+l)!}{(N-l-1)!}}, \qquad N = n+l+1. \qquad (56)$$

The susy-n superpotential as defined by (1) reads

$$W_n(r) = -\frac{\hbar}{\sqrt{2\mu}}[\frac{l+1}{r} - \frac{\lambda}{2} - \frac{n\lambda F(1)}{(2l+2)F(0)}], \qquad (57)$$

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whose special susy-0 case is

$$W_0(r) = \sqrt{\frac{\mu}{2}} \frac{e^2}{(l+1)\hbar} - \frac{(l+1)\hbar}{\sqrt{2\mu}r}. \tag{58}$$

In the above we used a similar short-hand notation for the confluent hypergeometric series (function) as in the previous section, namely

$$F(i) = F(-n+i, 2l+2+i; \lambda r), \qquad i = 0, 1, 2. \tag{59}$$

Further we calculate the partner potentials

$$
\begin{aligned}
V_n^-(r) &= V_l(r) - E_n = -\frac{e^2}{r} + \frac{l(l+1)\hbar^2}{2\mu r^2} + \frac{\mu e^4}{2\hbar^2(n+l+1)^2}, \\
V_n^+(r) &= -\frac{e^2}{r} + \frac{(l+1)(l+2)\hbar^2}{2\mu r^2} + \frac{\mu e^4}{2\hbar^2(n+l+1)^2} + I_n, \\
I_n &= \frac{n\lambda^2\hbar^2}{\mu(2l+2)F(0)} \Big[\frac{(1-n)}{(2l+3)}F(2) + \frac{nF^2(1)}{(2l+2)F(0)}\Big] \tag{60}
\end{aligned}
$$

Now in case of susy-0 partner potnatials we recover the shape invariance property embodied in the following relationships

$$
\begin{aligned}
V_0^-(r) &= -\frac{e^2}{r} + \frac{l(l+1)\hbar^2}{2\mu r^2} + \frac{\mu e^4}{2\hbar^2(l+1)^2} \equiv V_0^-(r, l), \\
V_0^+(r) &= -\frac{e^2}{r} + \frac{(l+1)(l+2)\hbar^2}{2\mu r^2} + \frac{\mu e^4}{2\hbar^2(l+1)^2} \equiv V_0^+(r, l) \\
V_0^+(r, l) &= V_0^-(r, l+1) + \frac{\mu e^4}{2\hbar^2}\Big[\frac{1}{(l+1)^2} - \frac{1}{(l+2)^2}\Big], \\
V_0^+(r, a_0) &= V_0^-(r, a_1) + R(a_1), \qquad a_o = l, \qquad a_1 = l+1, \\
R(a_1) &= \frac{\mu e^4}{2\hbar^2}\Big[\frac{1}{(l+1)^2} - \frac{1}{(l+2)^2}\Big].
\end{aligned}
\tag{61}
$$

Finally we calculate the partner potentials for the two lowest excited states. We find

$$V_1^+(r) = -\frac{e^2}{r} + \frac{(l+1)(l+2)\hbar^2}{2\mu r^2} + \frac{\mu e^4}{2\hbar^2(l+2)^2} + I_1,$$

$$I_1 = \frac{\lambda^2\hbar^2}{\mu(2l+2)^2}[1 - \frac{\lambda r}{(2l+2)}]^{-2}$$

$$(62)$$

and

$$V_2^+(r) = -\frac{e^2}{r} + \frac{(l+1)(l+2)\hbar^2}{2\mu r^2} + \frac{\mu e^4}{2\hbar^2(l+3)^2} + I_2,$$

$$I_2 = \frac{8\mu e^4 \beta_0}{(l+3)^2\hbar^2}(1 - 2\beta 0 z + \beta_0\beta_1 z^2)^{-1} \times$$

$$\times \quad [-\beta_1 + z\beta_0(1-\beta_1 z)^2(1 - 2\beta_0 z + \beta_0\beta_1 z^2)^{-1}],$$

$$\beta_0^{-1} = 2l+2, \qquad \beta_1^{-1} = \beta_0^{-1} + 1, \qquad z = \frac{2r}{(l+3)a} = \frac{2\mu e^4 r}{(l+3)\hbar^2}.$$

$$(63)$$

Finally we compute the eigenenergies

$$E_p^+ = E_{n+1+p}^- = E_{n+1+p} - E_n = \frac{\mu e^4}{2\hbar^2}[(n+l+1)^{-2} - (n+p+l+2)^{-2}], \qquad E_0^+ = E_{n+1}^-.$$

$$(64)$$

and the eigenfunctions using equations (6), (19), (52),

$$\chi_p^+(r) = \frac{1}{\sqrt{E_{n+1+p}^-}}A\chi_{n+1+p}^-(r), \qquad \chi_{n+1+p}^-(r) = \chi_{n+1+p}(r), \qquad p = 0, 1, 2, \ldots$$

$$(65)$$

6   The 1-dim Morse potential

In this section we analyze the supersymmetric aspects of the Morse potential, defined as

$$V(x) = A(e^{-2\alpha x} - 2e^{-\alpha x}), \tag{66}$$

which is an important model potential. It has a finite number of eigenstates. The notation for the constant A here should not be confused with the (ladder) operator A from section 2. The Schrödinger equation

$$\frac{d^2\psi(x)}{dx^2} + \frac{2\mu}{\hbar^2}(E - V(x))\psi(x) = 0, \tag{67}$$

can be rewritten as

$$\frac{d^2\psi}{d\xi^2} + \frac{1}{\xi}\frac{d\psi}{d\xi} + (-\frac{1}{4} + \frac{n + s + \frac{1}{2}}{\xi} - \frac{s^2}{\xi^2})\psi = 0. \tag{68}$$

where we use the notations

$$\xi = \frac{2\sqrt{2\mu A}}{\alpha\hbar}e^{-\alpha x}, \qquad s = \frac{\sqrt{-\mu E}}{\alpha\hbar}, \qquad n = \frac{\sqrt{2\mu A}}{\alpha\hbar} - (s + \frac{1}{2}). \tag{69}$$

Now we try the Ansatz

$$\psi(\xi) = Ce^{-\xi/2}\xi^s u(\xi), \tag{70}$$

such that the function $u(\xi)$ must satisfy the simpler equation

$$\xi u'' + (2s + 1 - \xi)u' + nu = 0, \tag{71}$$

having the solution in terms of the confluent hypergeometric function

$$u(\xi) = F(-n, 2s + 1; \xi), \tag{72}$$

so that finally the explicit solution obtains the form

$$\psi_n(\xi) = C\xi^s e^{-\xi/2}F(-n, 2s + 1; \xi), \tag{73}$$

with the energy spectrum

$$E_n = -A[1 - \frac{\alpha\hbar}{\sqrt{2\mu A}}(n + \frac{1}{2})]^2, \qquad n = 0, 1, 2, \ldots, < (\frac{\sqrt{2\mu A}}{\alpha\hbar} - \frac{1}{2}) \tag{74}$$

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and the quantum number n runs up to the maximum value.

Now employing the same procedure as before we find the susy-n super-potential $W_n(x)$, namely

$$W_n(x) = -\frac{\hbar}{\sqrt{2\mu}}[-\alpha s + \frac{1}{2}\alpha\xi + +\frac{n\alpha\xi F(1)}{(2s+1)F(0)}], \quad (75)$$

whose special and well known ground state $(n = 0)$ value is

$$W_0(x) = \sqrt{A} + \frac{\alpha}{2} - \sqrt{A}e^{-\alpha x}. \quad (76)$$

Again, we use here the short-hand notation for the confluent hypergeometric function

$$F(i) = F(-n + i, 2s + 1 + i; \xi), \qquad i = 0, 1, 2. \quad (77)$$

The starting shifted potential of (66) is

$$V_n^-(x) = V(x) - E_n = A(e^{-2\alpha x} - 2e^{-\alpha x}) + A[1 - \frac{\alpha\hbar}{\sqrt{2\mu A}}(n + \frac{1}{2})]^2, \quad (78)$$

and we get the isospectral partner potential

$$
\begin{aligned}
V_n^+(x) &= Ae^{-2\alpha x} - (2A - \sqrt{\frac{2A}{\mu}}\alpha\hbar)e^{-\alpha x} + A[1 - \frac{\alpha\hbar}{\sqrt{2\mu A}}(n + \frac{1}{2})]^2 + I_n, \\
I_n &= \frac{n\alpha^2\hbar^2\xi}{(2s+1)\mu F(0)}[F(1) + \frac{(1-n)\xi F(2)}{(2s+2)} + \frac{n\xi F^2(1)}{(2s+1)F(0)}].
\end{aligned}
\quad (79)
$$

The shape invariance is recovered for $n = 0$, namely the shifted starting potential is

$$V_0^-(x) = V(x) - E_0 = A(e^{-2\alpha x} - 2e^{-\alpha x}) + A(1 - \frac{\alpha\hbar}{2\sqrt{2\mu A}})^2 \equiv V_0^-(x; 2A), \quad (80)$$

and the isospectral partner potential

$$V_0^+(x) = Ae^{-2\alpha x} - (2A - \sqrt{\frac{2A}{\mu}}\alpha\hbar)e^{-\alpha x} + A(1 - \frac{\alpha\hbar}{2\sqrt{2\mu A}})^2 \equiv V_0^+(x; 2A), \quad (81)$$

where the notation $V_0^\pm(x; 2A)$ implies dependence on the coefficient $2A$ in the Morse potential (66). Then we find the shape invariance relationship

$$
\begin{aligned}
V_0^+(x; a_0) &= V_0^-(x; a_1) + R(a_1), \\
a_0 &= 2A, \qquad a_1 = 2A - \sqrt{\frac{2A}{\mu}}\alpha\hbar, \\
R(a_1) &= A(1 - \frac{\alpha\hbar}{2\sqrt{2A\mu}})^2
\end{aligned}
$$
(82)

The isospectral partner potentials of V_1^- and of V_2^- are calculated in a straightforward manner:

$$
\begin{aligned}
V_1^+(x) &= Ae^{-2\alpha x} - (2A - \sqrt{\frac{2A}{\mu}}\alpha\hbar)e^{-\alpha x} + A(1 - \frac{3\alpha\hbar}{2\sqrt{2\mu A}})^2 + I_1, \\
I_1 &= \frac{\alpha^2\beta_0\hbar\xi}{\mu}(1 - \beta_0\xi)^{-1}[1 + \beta_0\xi(1 - \beta_0\xi)^{-1}]
\end{aligned}
$$
(83)

and

$$
\begin{aligned}
V_2^+(x) &= Ae^{-2\alpha x} - (2A - \sqrt{\frac{2A}{\mu}}\alpha\hbar)e^{-\alpha x} + A(1 - \frac{5\alpha\hbar}{2\sqrt{2\mu A}})^2 + I_2, \\
I_2 &= 2\alpha^2\beta_0\hbar^2\xi(1 - 2\beta_0\xi + \beta_0\beta_1\xi^2)^{-1} \times \\
&\quad \times [1 - \beta_1\xi + 2\beta_0\xi(1 - 2\beta_0\xi + \beta_0\beta_1\xi^2)^{-1}(1 - \beta_1\xi)^2]
\end{aligned}
$$
(84)

where we use the notation for the parameters β_0, β_1,

$$\beta_0 = 2s + 1, \qquad \beta_1^{-1} = \beta_0^{-1} + 1. \quad (85)$$

Finally the normalized wavefunctions of the partner potential $V_n^+(x)$ are calculated easily

$$\psi_p^+ = \frac{1}{\sqrt{E_{n+1+p}^-}}\hat{A}\psi_{n+1+p}^-, \qquad \psi_{n+1+p}^- = \psi_{n+1+p}, \qquad E_{n+1+p}^- = E_{n+1+p} - E_n,$$

(86)

where again $p = 0, 1, 2, \ldots$, and $\hat{A}$ here is operator defined in (6) with the superpotential given in (75).

7 The 1-dim Pöschl-Teller type I potential

The 1-dim Pöschl-Teller potential is defined as (Flügge 1971)

$$V(x) = \frac{1}{2}V_0\left[\frac{A(A-1)}{\sin^2\alpha x} + \frac{B(B-1)}{\cos^2\alpha x}\right],$$

(87)

where

$$V_0 = \frac{\hbar^2\alpha^2}{\mu}, \qquad 0 \le \alpha x \le \frac{\pi}{2}.$$

(88)

The solution is well known

$$\psi_n(x) = C\sin^A\alpha x \cos^B\alpha x F(-n, A + B + n; \sin^2\alpha x),$$

(89)

with the energy spectrum

$$E_n = \frac{1}{2}V_0(A + B + 2n)^2, \qquad n = 0, 1, 2, \ldots.$$

(90)

The susy-n superpotential is

$$
\begin{aligned}
W_n(x) &= -\frac{\alpha\hbar}{\sqrt{2\mu}}\left[A\cot\alpha x - B\tan\alpha x - \frac{2n(A + B + n)\sin\alpha x \cos\alpha x F_n(1)}{(A + \frac{1}{2})F_n(0)}\right] \\
W_0(x) &= -\frac{\alpha\hbar}{\sqrt{2\mu}}(A\cot\alpha x - B\tan\alpha x)
\end{aligned}
$$

(91)

where we employ the short-hand notation for the hypergeometric series (Abramowitz and Stegun 1965 p.555)

$$F_n(i) = F(-n + i, A + B + n + i, A + \frac{1}{2} + i; \sin^2 \alpha x). \qquad (92)$$

The starting shifted potential is

$$V_n^-(x) = V(x) - E_n = \frac{1}{2}V_0[\frac{A(A-1)}{\sin^2 \alpha x} + \frac{B(B-1)}{\cos^2 \alpha x} - (A+B+2n)^2], \quad (93)$$

and the isospectral partner potential

$$
\begin{aligned}
V_n^+(x) &= \frac{1}{2}V_0[\frac{A(A+1)}{\sin^2 \alpha x} + \frac{B(B+1)}{\cos^2 \alpha x} - (A+B+2n)^2] + I_n, \\
I_n &= \frac{2V_0 n(A+B+n)}{(A+\frac{1}{2})F_n(0)}\{F_n(1)(1 - 2\sin^2 \alpha x) + \\
&+ [\frac{2(-n+1)(A+B+n+1)F_n(2)}{(A+3/2)} + \frac{2n(A+B+n)F_n^2(1)}{(A+1/2)F_n(0)}]\sin^2 \alpha x \cos^2 \alpha x\}.
\end{aligned}
$$
$$(94)$$

In case $n = 0$ we again recover the shape invariance property

$$
\begin{aligned}
V_0^-(x) &= \frac{1}{2}V_0[\frac{A(A-1)}{\sin^2 \alpha x} + \frac{B(B-1)}{\cos^2 \alpha x} - (A+B)^2] \equiv V_0^-(x, A, B), \\
V_0^+(x) &= \frac{1}{2}V_0[\frac{A(A+1)}{\sin^2 \alpha x} + \frac{B(B+1)}{\cos^2 \alpha x} - (A+B)^2] \equiv V_0^+(x, A, B), \\
V_0^+(x, \{a_0\}) &= V_0^-(x, \{a_1\}) + R(\{a_1\}), \\
\{a_0\} &= \{A, B\}, \qquad \{a_1\} = \{A+1, B+1\}, \qquad R(\{a_1\}) = V_0(A+B+2).
\end{aligned}
$$
$$(95)$$

The isospectral partner potentials for the lowest two excited states $n = 1, 2$ are

$$V_1^+(x) = \frac{1}{2}V_0[\frac{A(A+1)}{\sin^2 \alpha x} + \frac{B(B+1)}{\cos^2 \alpha x} - (A+B+2)^2] + I_1,$$

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$$I_1 = \frac{2V_0(A+B+1)}{(A+\frac{1}{2})[1 - \frac{(A+B+1)}{(A+\frac{1}{2})}\sin^2\alpha x]} \times$$

$$\times \quad \{1 - 2\sin^2\alpha x + \frac{2(A+B+1)\sin^2\alpha x\cos^2\alpha x}{(A+\frac{1}{2})[1 - \frac{(A+B+1)}{(A+\frac{1}{2})}\sin^2\alpha x]}\}$$

$$\tag{96}$$

and

$$V_2^+(x) = \frac{1}{2}V_0[\frac{A(A+1)}{\sin^2\alpha x} + \frac{B(B+1)}{\cos^2\alpha x} - (A+B+4)^2] + I_2,$$

$$I_2 = \frac{4V_0(A+B+2)}{(A+\frac{1}{2})F_2(0)}\{F_2(1)(1 - 2\sin^2\alpha x) +$$

$$+ \quad [-\frac{2(A+B+3)}{(A+3/2)} + \frac{4(A+B+2)F_2^2(1)}{(A+\frac{1}{2})F_2(0)}]\sin^2\alpha x\cos^2\alpha x\}$$

$$\tag{97}$$

The procedure to obtain the normalized eigenfunctions of the isospectral partner potential V_n^+ is the same as in the previous sections and in general it is explained in the section 2.

8   The 1-dim box potential

The 1-dim box potential is defined as

$$V(x) = 0 \quad \text{for} - L/2 \le x \le +L/2, \quad \text{and} \quad V(x) = \infty \quad \text{otherwise.} \tag{98}$$

The solutions of the Schrödinger equation

$$\frac{\hbar^2}{2\mu}\frac{d^2\psi}{dx^2} + (E - V(x))\psi = 0, \tag{99}$$

are

$$\psi_n(x) = \sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}, \qquad \text{odd} \quad n = 1, 3, , \ldots$$

$$\psi_n(x) \;=\; \sqrt{\frac{2}{L}}\,\sin\frac{n\pi x}{L}, \qquad \text{even}\quad n = 2, 4, \ldots$$

(100)

with the energy spectrum

$$E_n = \frac{\hbar^2 \pi^2}{2\mu L^2}n^2, \qquad n = 1, 2, \ldots.$$

(101)

The shifted starting potential is

$$V_n^-(x) = -E_n \quad \text{for} - L/2 \le x \le +L/2, \quad \text{and} \quad V(x) = \infty \quad \text{otherwise},$$

(102)

the superpotential is

$$W_n(x) = -\frac{\hbar}{\sqrt{2\mu}}\frac{\psi_n'}{\psi_n},$$

(103)

and therefore using equation (14) we get the partner potential

$$
\begin{aligned}
V_n^+(x) &= -E_n + \frac{\hbar^2}{\mu}\frac{1}{\cos^2 k_n x}, & \text{odd}, \\
V_n^+(x) &= -E_n + \frac{\hbar^2}{\mu}\frac{1}{\sin^2 k_n x}, & \text{even}, \\
k_n &= \frac{n\pi}{L}, \qquad E_n = \frac{\hbar^2}{2\mu}k_n^2, & n = 1, 2, 3, \ldots
\end{aligned}
$$

(104)

Thus the isospectral partner potential to the 1-dim box potential is a special case of the Pöschl-Teller type I potential, defined in (87). The eigenenergies for $V_n^+(x)$ are

$$E_p^+ = E_{n+1+p} - E_n = \frac{\hbar^2 \pi^2}{2\mu L^2}[(n+1+p)^2 - n^2], \qquad p = 0, 1, 2, \ldots$$

(105)

and the eigenfunctions are easily obtained by applying the formula (19).

9 Discussion and conclusions

In this paper we have applied the supersymmetric formalism of the (nonrelativistic) quantum mechanics, introduced and explained in (Robnik 1997), to the higher excited states of five different specific exactly solvable potentials, namely (i) 3-dim (spherically symmetric) harmonic oscillator, (ii) 3-dim (isotropic) Kepler problem, (iii) Morse potential, (iv) Pöschl-Teller type I potential, and (v) the 1-dim box potential. In all cases except in (v) we get new classes of isospectral partner potentials which thus also fall into the class of exactly solvable potentials although they can be and typically are quite complex. The most important case of the 1-dim harmonic oscillator was treated in detail in paper (I) (Robnik 1997).

We have also shown, in section (3), that for higher excited states there is generally no shape invariance of the usual type, and it remains to be investigated if there are some other types of "shape invariance", generated by some other functional relationships between the starting and partner potentials, such that they would allow for an exact solution of the problem. At present we do not know any specific cases of susy-n shape invariance with $n > 0$. Further calculations for susy-n partner potentials isospectral to some other well known exactly solvable potentials, not analyzed in this paper and in (I), remain as a future project.

Acknowledgements

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